An epistemic logic for update semantics – Ⅰ *

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For convenience, “if…then…” and “if and only if” are usually abbreviated to $\Rightarrow$ and $\Leftrightarrow$, respectively.

First we give some basic facts of set theory, which are often used in this paper:

Lemma 1.1 Let S, T and U be sets. Then

(1) $T \subseteq S \Leftrightarrow S \cap T = T$.

(2) Let $T \subseteq S$. Then $S - T = S \Leftrightarrow T = \varnothing$.

(3) Let $T \subseteq S$. Then $S - T = \varnothing \Leftrightarrow S = T$.

(4) Let $T \subseteq S$. $S - (S - T) = T$.

(5) $S - (T - U) = (S - T) \cup (S \cap U)$.

(6) Let T, $U \subseteq S$. Then $(S - U) \cup T = S \Leftrightarrow U \subseteq T$.

(7) Let $T \subseteq S$. Then $(S - T) \cup T = S$ (by (6)).

(8) $S - (T \cup U) = (S - T) \cap (S - U)$. ┤

Let PV := $\{p_1, \ldots, p_n, \ldots\}$ be a countable set of propositional variables.

Definition 1.2

(1) A *language* L is a set of formulas φ, given by the following rules:

$\varphi := p \mid \neg\varphi \mid (\varphi \vee \psi) \mid K\varphi$ where $p \in$ PV.

(2) For all $\varphi, \psi \in L$, we introduce the following abbreviations:

$(\varphi \wedge \psi) := \neg(\neg\varphi \vee \neg\psi)$

$(\varphi \to \psi) := (\neg\varphi \vee \psi)$,

$(\varphi \leftrightarrow \psi) := ((\varphi \to \psi) \wedge (\psi \to \varphi))$.

$\top := (p_1 \vee \neg p_1)$,

$\bot := \neg\top$.

(3) Given any $\varphi \in L$, φ is called a *modal formula* $\Leftrightarrow$ φ contains K. ┤

Kφ represents intuitively that the current agent knows φ.

If not especially mention henceforth, we always use *metavariables* p, q, … (with or without

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subscripts) as formulas in PV, use *metavariables* φ, ψ, … (with or without subscripts) as formulas in L, and use Φ, Ψ, … (with or without the subscript) as formula sets, namely, subsets of L.

As usual, we will omit the outside parentheses of a formula and the inside parentheses subject to the convention $\neg$, K, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ are taken in this order of priority.

Let X and Y be two sets, f: $X \rightarrow Y$ a function from X into Y, and $Y^X := \{f \mid f: X \rightarrow Y\}$. Let W be a set, and $\wp(W)$ the *power set* of W. For convenience, below, for any $S \subseteq W$ and $f \in \wp(W)^{\wp(W)}$, we use Sf for $f(S)$.

Definition 1.3

(1) A *frame* for L is a pair (W, R) such that $W \neq \varnothing$ is a set of possible worlds, and R is a binary relation on $\wp(W)$: $R \subseteq \wp(W) \times \wp(W)$.

(2) A *model* for L is a tuple $(W, R, V, \|\,.\,\|)$ such that (W, R) is a frame, and

V: PV $\rightarrow \wp(W)$, and

$\|\,.\,\|$: PV $\rightarrow \wp(W)^{\wp(W)}$ such that $S\,\|\,p\,\| = S \cap V(p)$ for all $S \subseteq W$ and $p \in$ PV. ⊣

If not especially mention henceforth, we always use *metavariables* w, u, v, …, possibly with subscripts, as elements of W, and use *metavariables* S, T, U, …, possibly with subscripts, as subsets of W.

Definition 1.4 (Truth Definition) Let φ and ψ be not modal formulas.

(1) $V(\neg\varphi) = W - V(\varphi)$,

(2) $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$. ⊣

Lemma 1.5 Let $(W, R, V, \|\,.\,\|)$ be a model, φ and ψ be not modal formulas. Then

(1) $V(\bot) = \varnothing$, $V(\top) = W$.

(2) $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$.

(3) $V(\varphi) \cap V(\varphi \rightarrow \psi) \subseteq V(\psi)$.

(4) $V(\varphi \rightarrow \psi) = W \Leftrightarrow V(\varphi) \subseteq V(\psi)$.

(5) $V(\varphi \leftrightarrow \psi) = W \Leftrightarrow V(\varphi) = V(\psi)$. ⊣

Definition 1.6 (Update) Let $(W, R, V, \|\,.\,\|)$ be a model and $S \subseteq W$. For all $\varphi, \psi \in L$,

(1) $S\,\|\,\neg\varphi\,\| = S - S\,\|\,\varphi\,\|$,

(2) $S\,\|\,\varphi \vee \psi\,\| = S\,\|\,\varphi\,\| \cup S\,\|\,\psi\,\|$,

(3) $S\,\|\,\mathrm{K}\varphi\,\| = S$ iff for each $T \subseteq W$, if $S \subseteq T$ and SRT, then $T\,\|\,\varphi\,\| = T$. [①] ⊣

For any φ, $\|\,\varphi\,\|$, if defined, can be called the *dynamic proposition* denoted by φ, and $V(\varphi)$, if defined, can be called the *static proposition* denoted by φ. $S\,\|\,\varphi\,\|$ represents intuitively the information state the current agent acquires by updating an information state S with φ, and SRT the indistinguishability of S and T the current agent considers.

Lemma 1.7 Let $(W, R, V, \|\,.\,\|)$ be a model, $S \subseteq W$ and let φ be not a modal formula. Then

(*) $S\,\|\,\varphi\,\| = S \cap V(\varphi)$.

Proof. We will prove (*) by induction on the complexity of φ.

If $\varphi \in$ PV, then by Definition 1.3(2), $S\,\|\,p\,\| = S \cap V(p)$.

If $\varphi = \neg\psi$, then

$S\,\|\,\neg\psi\,\| = S - S\,\|\,\psi\,\|$ (by Definition 1.6)

$= S - S \cap V(\psi)$ (by the induction hypothesis)

$= S \cap V(\neg\psi)$ (by some basic facts of set theory).

If $\varphi = \psi \vee \theta$, then

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① See Groeneveld and Veltman[1994], p. 232.

$S \parallel \psi \vee \theta \parallel = S \parallel \psi \parallel \cup \ S \parallel \theta \parallel$ (by Definition 1.6)

$= (S \cap V(\psi)) \ \cup \ (S \cap V(\theta))$ (by the induction hypotheses)

$= S \cap (V(\psi) \ \cup \ V(\theta))$ (by some basic facts of set theory)

$= S \cap V(\psi \vee \theta)$ (by Definition 1.6). ┤

Definition 1.8 Let $M = (W, R, V, \ \parallel . \parallel)$ be a model and $S \subseteq W$. M is an *epistemic model for update semantics* $\Leftrightarrow$ the following model conditions are satisfied: for all $\varphi, \psi \in L$,

(g) $S \parallel \varphi \parallel \subseteq S$.

(r) $S \parallel \mathrm{K}(\varphi \wedge \psi) \parallel = S \parallel \mathrm{K}\varphi \parallel \cap S \parallel \mathrm{K}\psi \parallel$.

(t) $S \parallel \mathrm{K}\varphi \parallel \subseteq S \parallel \varphi \parallel$.

(e) $S \parallel \neg\mathrm{K}\varphi \parallel \subseteq S \parallel \mathrm{K}\neg\mathrm{K}\varphi \parallel$.

(re) $\parallel \varphi \parallel = \parallel \psi \parallel \Rightarrow \parallel \mathrm{K}\varphi \parallel = \parallel \mathrm{K}\psi \parallel$.

In this paper Model is used as the class of all such models. ┤

Lemma 1.9 Let $(W, R, V, \ \parallel . \parallel) \in$ Model and $S \subseteq W$. For all $\varphi, \psi \in L$,

(1)(a) $S \parallel \varphi \wedge \psi \parallel = S \parallel \varphi \parallel \cap S \parallel \psi \parallel$.

(b) $S \parallel \varphi \parallel = S \parallel \psi \parallel = S \Leftrightarrow S \parallel \varphi \ \wedge \ \psi \parallel = S$.

(2) $S \parallel \varphi \rightarrow \psi \parallel = (S - S \parallel \varphi \parallel) \cup S \parallel \psi \parallel$.

(3) $S \parallel \varphi \rightarrow \psi \parallel = S \Leftrightarrow S \parallel \varphi \parallel \subseteq S \parallel \psi \parallel$ (by (2), Definition 1.8(g) and Lemma 1.1(6)).

(4) $S \parallel \varphi \leftrightarrow \psi \parallel = S \Leftrightarrow S \parallel \varphi \parallel = S \parallel \psi \parallel$ (by (1)(b) and (3)).

Proof. Verify (1). The proof of (a): We have

$S \parallel \varphi \wedge \psi \parallel = S \parallel \neg(\neg\varphi \vee \neg\psi) \parallel$ by Definition 1.2(2)

$= S - S \parallel \neg\varphi \vee \neg\psi \parallel$ by Definition 1.6(1)

$= S - (S \parallel \neg\varphi \parallel \cup \ S \parallel \neg\psi \parallel)$ by Definition 1.6(2)

$= (S - S \parallel \neg\varphi \parallel) \cap (S - S \parallel \neg\psi \parallel)$ by Lemma 1.1(8)

$= S \parallel \varphi \parallel \cap S \parallel \psi \parallel$ by Definition 1.6(1).

The proof of (b): Assume that $S \parallel \varphi \parallel = S \parallel \psi \parallel = S$. By (a), we have $S \parallel \varphi \ \wedge \ \psi \parallel = S$.

Assume that $S \parallel \varphi \ \wedge \ \psi \parallel = S$, then $S \parallel \varphi \parallel \cap S \parallel \psi \parallel = S$. By Definition 1.8(g), $S \parallel \varphi \parallel \subseteq S$. If $S \parallel \varphi \parallel = T \subset S$, then

$S \parallel \varphi \parallel \cap S \parallel \psi \parallel = T \cap S \parallel \psi \parallel \subseteq T \subset S$.

This is impossible. So $S \parallel \varphi \parallel = S$ and thus

$S \parallel \psi \parallel = S \parallel \varphi \parallel \cap S \parallel \psi \parallel = S \parallel \varphi \ \wedge \ \psi \parallel = S$.

Verify (2).

$S \parallel \varphi \rightarrow \psi \parallel = S \parallel \neg\varphi \vee \psi \parallel$ by Definition 1.2(2)

$= S \parallel \neg\varphi \parallel \cup \ S \parallel \psi \parallel$ by Definition 1.6(2)

$= (S - S \parallel \varphi \parallel) \cup \ S \parallel \psi \parallel$ by Definition 1.6(1). ┤

Definition 1.10 (Validity Definition) Let $M = (W, R, V, \ \parallel . \parallel) \in$ Model.

(1) Let $S \subseteq W$. S *supports* φ, denoted by $S \vDash \varphi$, $\Leftrightarrow S \parallel \varphi \parallel = S$;

otherwise, denoted by $S \nvDash \varphi$.

(2) φ is *valid* in M, denoted by $M \vDash \varphi$, $\Leftrightarrow$ for all $S \subseteq W$, $S \vDash \varphi$;

otherwise, φ is *invalid* in M, denoted by $M \nvDash \varphi$.

(3) φ is *valid*, denoted by $\vDash \varphi$, $\Leftrightarrow$ for all $M \in$ Model, $M \vDash \varphi$;

otherwise, φ is *invalid*, denoted by $\nvDash \varphi$. ┤

According to Definition 1.10, we have:

Lemma 1.11 Let $M = (W, R, V, \ \parallel . \parallel) \in$ Model and $S \subseteq W$.

(1) If $S \vDash \varphi$ and $S \vDash \varphi \rightarrow \psi$, then $S \vDash \psi$.

(2) $\vDash \varphi \to \psi$ and $\vDash \varphi \Rightarrow \vDash \psi$ (by (1)).

Proof. Verify (1). Let $S \| \varphi \| = S \| \varphi \to \psi \| = S$. By $S \| \varphi \to \psi \| = S$ and Lemma 1.9(3) ,

$S \| \varphi \| \subseteq S \| \psi \|$.

So by $S \| \varphi \| = S$ and Definition 1.8(g), we have $S \| \psi \| = S$. ⊣

Lemma 1.12

(1) φ is an instantiation of a tautology $\Rightarrow \vDash \varphi$.

(2) $\vDash$ K⊤.

(3) $\vDash \mathrm{K}\varphi \wedge \mathrm{K}\psi \leftrightarrow \mathrm{K}(\varphi \wedge \psi)$.

(4) $\vDash \mathrm{K}\varphi \to \varphi$.

(5) $\vDash \neg\mathrm{K}\varphi \to \mathrm{K}\neg\mathrm{K}\varphi$.

(6) For any $M \in$ Model, $M \vDash \varphi \leftrightarrow \psi \Rightarrow M \vDash \mathrm{K}\varphi \leftrightarrow \mathrm{K}\psi$.

Proof. Verify (1). Given any $(W, R, V, \| . \|) \in$ Model and $S \subseteq W$. By Lemma 1.11(2), it suffices to show that

① $S \vDash \varphi \vee \varphi \to \varphi$.

② $S \vDash \varphi \to \varphi \vee \psi$.

③ $S \vDash \varphi \vee \psi \to \psi \vee \varphi$.

④ $S \vDash (\psi \to \theta) \to \varphi \vee \psi \to \varphi \vee \theta$. [②]

By Lemma 1.9(3) and Definition 1.6, it is clear that ① - ③ hold.

Below we will verify ④. By Lemma 1.9(3), it suffices to show that

(a) $S \| \psi \to \theta \| \subseteq S \| \varphi \vee \psi \to \varphi \vee \theta \|$.

So by Lemma 1.9(2) and Definition 1.6, it suffices to show that

(b) $(S - S \| \psi \|) \cup S \| \theta \| \subseteq (S - (S \| \varphi \| \cup S \| \psi \|)) \cup S \| \varphi \| \cup S \| \theta \|$.

So by Lemma 1.1(8), it suffices to show that

(c) $(S - S \| \psi \|) \cup S \| \theta \| \subseteq ((S - S \| \varphi \|) \cap (S - S \| \psi \|)) \cup S \| \varphi \| \cup S \| \theta \|$.

Given any $w \in (S - S \| \psi \|) \cup S \| \theta \|$. If $w \in S \| \theta \|$, then it is clear that (c) holds. So let $w \in (S - S \| \psi \|)$, then $w \in S$. If $w \in S \| \varphi \|$, then it is clear that (c) holds as well. Hence we let $w \notin S \| \varphi \|$. Thus $w \in (S - S \| \varphi \|)$ and hence $w \in ((S - S \| \varphi \|) \cap (S - S \| \psi \|))$. So (c) holds.

Verify (2). Let $(W, R, V, \| . \|)$ and $S \subseteq W$. Then by Lemma 1.1(1) and Lemma 1.7(*),

for each $T \subseteq W$, if $S \subseteq T$ and SRT, then $T \| \top \| = T$.

Hence $S \vDash$ K⊤ by Definition 1.6(3).

Verify (3). Let $(W, R, V, \| . \|)$ and $S \subseteq W$. We have

$S \| \mathrm{K}(\varphi \wedge \psi) \| = S \| \mathrm{K}\varphi \| \cap S \| \mathrm{K}\psi \|$ by Definition 1.8(r)

$= S \| \mathrm{K}\varphi \wedge \mathrm{K}\psi \|$ by Lemma 1.9(1)(a).

So by Lemma 1.9(4), $S \vDash \mathrm{K}\varphi \wedge \mathrm{K}\psi \leftrightarrow \mathrm{K}(\varphi \wedge \psi)$.

Verify (4). Let $(W, R, V, \| . \|)$ and $S \subseteq W$. By Definition 1.8(t),

$S \| \mathrm{K}\varphi \| \subseteq S \| \varphi \|$.

So by Lemma 1.9(3), $S \vDash \mathrm{K}\varphi \to \varphi$.

Verify (5). Let $(W, R, V, \| . \|)$ and $S \subseteq W$. By Definition 1.8(e),

$S \| \neg\mathrm{K}\varphi \| \subseteq S \| \mathrm{K}\neg\mathrm{K}\varphi \|$.

So by Lemma 1.9(3), $S \vDash \neg\mathrm{K}\varphi \to \mathrm{K}\neg\mathrm{K}\varphi$.

Verify (6). Given any $M = (W, R, V, \| . \|) \in$ Model. Assume that $M \vDash \varphi \leftrightarrow \psi$, then by Lemma 1.9(4),

(a) $S \| \varphi \| = S \| \psi \|$ for all $S \subseteq W$.

[②] See LI Xiaowu[2005], p.66.

It is easy to see that the domain of $\| \varphi \|$ and that of $\| \psi \|$ are the same, so $\| \varphi \| = \| \psi \|$ by (a), and thus $\| K\varphi \| = \| K\psi \|$ by Definition 1.8(re), so

(b) $S \| K\varphi \| = S \| K\psi \|$ for all $S \subseteq W$.

So by Lemma 1.9(4), $S \vDash K\varphi \leftrightarrow K\psi$ for all $S \subseteq W$. ⊣

Definition 1.13 An *epistemic system* **ES5** is defined as follows:

(Taut) all instantiations of tautologies,

(N_K) $K\top$,

(R_K) $K\varphi \wedge K\psi \leftrightarrow K(\varphi \wedge \psi)$,

(T_K) $K\varphi \rightarrow \varphi$,

(5_K) $\neg K\varphi \rightarrow K\neg K\varphi$,

(MP) $\varphi, \varphi \rightarrow \psi \,/\, \psi$ (*modus ponens*),

(RE_K) $\varphi \leftrightarrow \psi \,/\, K\varphi \leftrightarrow K\psi$. ⊣

Definition 1.14 φ is a *theorem* of **ES5**, denoted as $\vdash \varphi$, means that which means φ has a formal proof in **ES5**, that is, there are formulas $\varphi_1, \ldots, \varphi_n$ such that for every $1 \leq i \leq n$, φ_i is an axiom of **ES5** or φ_i is obtained from some formulas in front of it by a rule of **ES5**.

By $\nvdash \varphi$ we mean that φ is not a theorem of **ES5**. ⊣

As usual, it is easy to prove that

Lemma 1.15 The following are a theorem and a derived rule of **ES5**:

(1) $K\varphi \rightarrow KK\varphi$,

(2) $\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \varphi \,/\, K\varphi_1 \wedge \ldots \wedge K\varphi_n \rightarrow K\varphi$. ⊣

By Lemma 1.12, we have

Theorem 1.16 (Soundness Theorem) Every theorem of **ES5** is valid in all elements of Model. ⊣

Below we will prove the completeness of **ES5**. We first give the following necessary definitions and lemmas preliminary to doing it.

Definition 1.17 Let w be a formula set.

(1) w is *consistent* ⇔ for all finite subset $u \subseteq w$, $\nvdash \neg\wedge u$.

(2) w is *maximal* ⇔ for every $\varphi \in L$, $\varphi \in w$ or $\neg\varphi \in w$.

(3) w is *maximal consistent* ⇔ w is maximal and consistent. ⊣

As usual, it is easy to prove the following two lemmas.

Lemma 1.18 Let w be a maximal consistent set. Then

(1) $\neg\varphi \in w \Leftrightarrow \varphi \notin w$.

$\varphi \wedge \psi \in w \Leftrightarrow \varphi \in w$ and $\psi \in w$.

$\varphi \vee \psi \in w \Leftrightarrow \varphi \in w$ or $\psi \in w$.

$\varphi \in w$ and $\vdash \varphi \rightarrow \psi \Rightarrow \psi \in w$.

$\varphi \in w$ and $\varphi \rightarrow \psi \in w \Rightarrow \psi \in w$.

(2) $\vdash \varphi \Rightarrow \varphi \in w$. ⊣

Lemma 1.19 Let W be the set of all maximal consistent sets and $| \varphi | := \{w \in W \mid \varphi \in w\}$. Then

(1) $| \neg\varphi | = W - | \varphi |$.

$| \varphi \wedge \psi | = | \varphi | \cap | \psi |$.

$| \varphi \vee \psi | = | \varphi | \cup | \psi |$.

$| \bot | = \varnothing$, $| \top | = W$.

(2) $| \varphi | \cap | \varphi \rightarrow \psi | \subseteq | \psi |$.

(3) $| \varphi \to \psi | = W \Leftrightarrow | \varphi | \subseteq | \psi | \Leftrightarrow \vdash \varphi \to \psi$.

(4) $| \varphi \leftrightarrow \psi | = W \Leftrightarrow | \varphi | = | \psi | \Leftrightarrow \vdash \varphi \leftrightarrow \psi$.

(5)(**Lindenbaum Lemma**) Let w be a consistent set. Then there is some $u \in W$ such that $w \subseteq u$.

(6) If $\nvdash \varphi$, then there is some $u \in W$ such that $\varphi \notin u$. ⊣

Definition 1.20 The *canonical model* is a tuple $(W, R, V, \| . \|)$ such that:

$W = \{w \mid w \text{ is maximal consistent}\}$,

$SRT \Leftrightarrow$ for all $\varphi \in L$, $S \subseteq T$ and $S \subseteq | \mathrm{K}\varphi | \Rightarrow T \subseteq | \varphi |$ for all $S, T \subseteq W$.

$V(\varphi) = | \varphi |$ for all $\varphi \in L$.

$S \| \varphi \| = S \cap | \varphi |$ for all $S \subseteq W$ and $\varphi \in L$. ⊣

Lemma 1.21 Let $M = (W, R, V, \| . \|)$ be the canonical model and $S \subseteq W$. Then

(1) $S \| \neg\varphi \| = S - S \| \varphi \|$.

(2) $S \| \varphi \vee \psi \| = S \| \varphi \| \cup S \| \psi \|$.

(3) $S \| \mathrm{K}\varphi \| = S \Leftrightarrow$ for each $T \subseteq W$, if $S \subseteq T$ and SRT, then $T \| \varphi \| = T$.

Proof. Verify (1). It is easy to see that

$$S \| \neg\varphi \| = S \cap | \neg\varphi | = S - S \cap | \varphi | = S - S \| \varphi \|.$$

Verify (2). It is easy to see that

$$S \| \varphi \vee \psi \| = S \cap | \varphi \vee \psi | = S \cap (| \varphi | \cup | \psi |)$$
$$= (S \cap | \varphi |) \cup (S \cap | \psi |) = S \| \varphi \| \cup S \| \psi \|.$$

Verify (3). First we have

(a) $S \| \varphi \| = S$ iff $S \cap | \varphi | = S$ iff $S \subseteq | \varphi |$.

So it suffices to show:

(b) $S \subseteq | \mathrm{K}\varphi | \Leftrightarrow$ for each $T \subseteq W$, if $S \subseteq T$ and SRT, then $T \subseteq | \varphi |$

'$\Rightarrow$': Straightforward.

'$\Leftarrow$': Assume that $S \nsubseteq |\mathrm{K}\varphi|$. We will show that

(c) there is some $T \subseteq W$ such that $S \subseteq T$ and SRT and $T \nsubseteq | \varphi |$.

By $S \nsubseteq | \mathrm{K}\varphi |$, there is some $w \in S$ such that $\mathrm{K}\varphi \notin w$. We will show that

(d) there is some $u \in W$ such that $\mathrm{K}^{-}w \subseteq u$ and $\varphi \notin u$, where $\mathrm{K}^{-}w := \{\varphi \mid \mathrm{K}\varphi \in w\}$.

Hypothesize that $\mathrm{K}^{-}w \cup \{\neg\varphi\}$ is not consistent, then there are $\varphi_1, \dots, \varphi_n \in \mathrm{K}^{-}w$ such that

$$\vdash \varphi_1 \wedge \dots \wedge \varphi_n \to \varphi.$$

So by Lemma 1.15(2),

$$\vdash \mathrm{K}\varphi_1 \wedge \dots \wedge \mathrm{K}\varphi_n \to \mathrm{K}\varphi.$$

By $\varphi_1, \dots, \varphi_n \in \mathrm{K}^{-}w$, we have $\mathrm{K}\varphi_1, \dots, \mathrm{K}\varphi_n \in w$, so $\mathrm{K}\varphi \in w$ by Lemma 1.18(1), contradicting that $\mathrm{K}\varphi \notin w$. So $\mathrm{K}^{-}w \cup \{\neg\varphi\}$ is consistent, and thus (d) holds by Lemma 1.19(5).

Let $T = S \cup \{u\}$. Then $S \subseteq T$. By (d), it is easy to see that $T \nsubseteq | \varphi |$.

To prove (c), it suffices to show that SRT. Given any $\psi \in L$ such that $S \subseteq T$ and $S \subseteq | \mathrm{K}\psi |$. It suffices to show that $T \subseteq | \psi |$. Given any $v \in T$. By $T = S \cup \{u\}$, we consider the following cases:

Case 1: $v \in S$. Then $\mathrm{K}\psi \in v$ by $S \subseteq | \mathrm{K}\psi |$, by $\psi \in v$ by Axiom $\mathrm{T_K}$.

Case 2: $v = u$. Hypothesize that $\psi \notin v$. Then $\psi \notin u$. So $\psi \notin \mathrm{K}^{-}w$ by (d), and thus $\mathrm{K}\psi \notin w$. On the other hand, $\mathrm{K}\psi \in w$ by $w \in S$ and $S \subseteq | \mathrm{K}\psi |$, and hence we have a contradiction. ⊣

Lemma 1.22 Let M be the canonical model. Then $M \in$ Model.

Proof. Let $M = (W, R, V, \| . \|)$ be the canonical model and $S \subseteq W$. It suffices to show that the model conditions in Definition 1.8 hold.

Verify (g). By Definition 1.20, $S \| \varphi \| = S \cap | \varphi | \subseteq S$.

Verify (r). We have,

$$
\begin{aligned}
S \| K(\varphi \wedge \psi) \| &= S \cap | K(\varphi \wedge \psi) | && \text{by Definition 1.20} \\
&= S \cap (| K\varphi \wedge K\psi |) && \text{by Axiom } R_K \text{ and Lemma 1.19(4)} \\
&= S \cap (| K\varphi | \cap | K\psi |) && \text{by Lemma 1.19(1)} \\
&= S \| K\varphi \| \cap S \| K\psi \| && \text{by Definition 1.20.}
\end{aligned}
$$

Verify (t). We have,

$$
\begin{aligned}
S \| K\varphi \| &= S \cap | K\varphi | && \text{by Definition 1.20} \\
&\subseteq S \cap | \varphi | && \text{by Axiom } T_K \text{ and Lemma 1.19(3)} \\
&= S \| \varphi \| && \text{by Definition 1.20.}
\end{aligned}
$$

Verify (e). We have,

$$
\begin{aligned}
S \| \neg K\varphi \| &= S \cap | \neg K\varphi | && \text{by Definition 1.20} \\
&\subseteq S \cap | K\neg K\varphi | && \text{by Axiom } \mathbf{5}_K \text{ and Lemma 1.19(3)} \\
&= S \| K\neg K\varphi \| && \text{by Definition 1.20.}
\end{aligned}
$$

Verify (re). Assume that $\| \varphi \| = \| \psi \|$. Then by Definition 1.20, $W \| \varphi \| = W \| \psi \|$, so

$$W \cap | \varphi | = W \cap | \psi |.$$

So $| \varphi | = | \psi |$. By Lemma 1.19(4) and Rule RE_K, $| K\varphi | = | K\psi |$, and thus

$$S \cap | K\varphi | = S \cap | K\psi | \text{ for all } S \subseteq W.$$

Hence by Definition 1.20,

$$S \| K\varphi \| = S \| K\psi \| \text{ for all } S \subseteq W.$$

It is easy to see that the domain of $\| K\varphi \|$ and that of $\| K\psi \|$ are the same, so $\| K\varphi \| = \| K\psi \|$. ⊣

Theorem 1.23 (Completeness Theorem) Every formula valid in all elements of Model is a theorem of **ES5**.

Proof. It suffices to show that

$$\nvdash \varphi \Rightarrow \text{there is some } M \in \text{Model such that } M \nvDash \varphi.$$

Let $M = (W, R, V, \| . \|)$ is the canonical model for **ES5**. Assume that $\nvdash \varphi$. By Lemma 1.19(6), there is some $u \in W$ such that $\varphi \notin u$, so $u \notin | \varphi |$, and thus

$$W \| \varphi \| = W \cap | \varphi | = | \varphi | \neq W.$$

Hence $W \nvDash \varphi$, and thus $M \nvDash \varphi$.

By the previous lemma, $M \in$ Model. ⊣

We conclude this paper by the following remarks: if we delete the phrase "$S \subseteq T$ and" in Definition 1.6(3), then the results above still hold.

References

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