Abstract—In this paper, we apply the induced generalised ordered weighted averaging (IGOWA) operator into the Dempster-Shafer (D-S) theory to define a model of decision making under ambiguity. Moreover, we reveal some properties of our new model. Finally, we validate our model by solving the Ellsberg paradox and the Machina paradoxes successfully.

I. INTRODUCTION

Generally speaking, aggregation operators are an important topic in decision making [1]–[8]. One of the most important aggregation operators is the ordered weighted averaging (OWA) operator [1]. It has lots of extensions [2], [9]–[11], such as the induced ordered weighted averaging (IOWA) operator [9], the ordered weighted geometric (OWG) operator [10], the induced ordered weighted geometric (IOWG) operator [12], and the generalized OWA (GOWA) operator [11].

To make decisions under ambiguity [13], [14], the OWA operator and its extensions such as the IOWA and IOWG operators are applied into D-S theory [5], [15]–[21], which is an effective tool for modeling belief about the uncertainty of ambiguity. However, there does not exist any unified formwork of these two models. Fortunately, Merigó et al. [2] introduce the induced generalised OWA (IGOWA) operator, which is a unified model of the IOWA operator and the IOWG operator. More specifically, the IGOWA operator [2] is an extension of the GOWA operator [11] by combining the reordering step with other mechanisms. And the GOWA operator is an extension of the OWA operator because it is a unified framework of the OWA operator, the OWG operator, and generalised means (GM).

So, since the IGOWA operator is a unified framework of different operators, in this paper we will apply it into D-S theory to establish a unified framework for decision making under ambiguity. We call our method DS-IGOWA operator. Moreover, in addition to analysing our model theoretically, we will also validate our model by solving the Ellsberg paradox [22] and the Machina paradoxes [23]. These paradoxes’ outcomes have been confirmed by the psychological experiments [22], [24], which have challenged various existing utility based models [25]. So, successfully solving these two paradoxes means our work is a very significant model for decision making under ambiguity.

The rest of this paper is organised as follows. Section 2 briefs the framework of D-S theory and the IGOWA operator. Section 3 introduces our DS-IGOWA operator and reveals some properties of the DS-IGOWA operator. Section 4 solves the Ellsberg paradox and the Machina paradoxes. Finally, Section 5 concludes the paper with the future work.

II. PRELIMINARIES

This section will recap D-S theory and the induced generalised OWA (IGOWA) operator.

D-S theory [15], [16], [26] can model not only the precise probability belief when the evidence is sufficient, but also the ambiguous probability belief when the evidence is insufficient, even missing [17]. Formally, we have:

Definition 1. Let X be a finite set of mutually disjoint atomic elements, called a frame of discernment, and 2^X be the set of all the subsets of X. Then a basic probability assignment, or called a mass function, is a mapping of m : 2^X → [0,1], which satisfies m(∅) = 0, and \(\sum_{A \subseteq X} m(A) = 1\). Subset A (⊂ X) satisfying m(A) > 0 is called a focal element of mass function m.

The induced generalised OWA (IGOWA) operator [2] is formally defined as follows:

Definition 2. Associated with weighting vector \(W = (w_1, \ldots, w_n)\) (where \(w_i \in [0,1]\) and \(\sum_{i=1}^{n} w_i = 1\)) and a parameter \(\lambda \in (-\infty, \infty)\), an IGOWA operator of dimension \(n\) is a mapping of \(U : ([0,1] \times \mathbb{R})^n \rightarrow \mathbb{R}\), given by:

\[
U((v_1, u_1), \ldots, (v_n, u_n)) = \left(\sum_{i=1}^{n} w_i b_i^\lambda\right)^{1/\lambda},
\]

where \(b_i\) is \(u_j\) such that \((v_j, u_j)\) has the \(i\)-th largest order-inducing variable \(v_j\), and \(u_j\) is the argument variable.

For example, suppose \(\{(3, 10), (5, 12), (1, 11), (7, 8)\}\) is the collection of arguments with their respective order-inducing variables, the weighting vector \(W = (0.5, 0.1, 0.2, 0.2)\), and \(\lambda = 1\), then, by formula (1), the aggregated result is: 0.5 \(\times\) 8 + 0.1 \(\times\) 12 + 0.2 \(\times\) 10 + 0.2 \(\times\) 11 = 9.4. Here, \(b_1 = 8\) because the inducing variable \(v_4 = 7\) of \((v_4, u_4) = (7, 8)\) is the largest value among all inducing variables, i.e., \{3, 5, 1, 7\}.

Specifically, the weighting vector reflects the weight of each argument in the aggregation operator, which can be obtained by different methods [27].

III. DECISION MAKING

In this section, we will apply the IGOWA operator into D-S theory to develop a new decision rule for decision making under ambiguity.
Suppose in a decision problem, we have a set of choices \( \{C_1, \cdots, C_n\} \) with a set of outcomes \( \{O_1, \cdots, O_n\} \), and \( u_{j,k} \) is the utility to the decision maker if choice \( C_j \) is selected and the outcome is \( O_k \). The belief about the outcome is represented by each focal element’s mass function value \( m(A_i) \). If the decision maker selects choice \( C_j \) and the focal element is \( A_k \), he can get the utility collection, called \( B_{j,k} \). Hence, \( B_{j,k} = \{(v_{j,k}, u_{j,k}) | O_k \in A_k\} \), where \( v_{j,k} \) is the order-inducing variable corresponding with utility \( u_{j,k} \). The decision maker’s objective is to select the choice with the biggest expected utility, which we define as follows:

**Definition 3.** A DS-IGOWA operator is a mapping of \([0, 1]^h \times \mathbb{R}^{nk} \to \mathbb{R} \), given by:

\[
E(C_j) = \sum_{k=1}^{h} m(A_k)U(B_{j,k})
\]

where \( m(A_k) \) is the mass function for focal element \( A_k \), \( h \) is the number of focal elements, \( B_{j,k} \) is the utility collection given by \( B_{j,k} = \{(v_{j,k}, u_{j,k}) | O_k \in A_k\} \), and \( U \) is the IGOWA operator given by formula (1), i.e.,

\[
U(B_{j,k}) = \left( \sum_{i=1}^{n_k} w_i b_{i,k}^\lambda \right)^{1/\lambda},
\]

where \( w_i \) is the weighting vector of focal element \( A_k \) such that \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n_k} w_i = 1 \), \( \lambda \in (-\infty, \infty) \), \( b_{i,k} \) is the utility of \((v_{j,k}, u_{j,k})\) having the \( k \)-th largest order-inducing variable \( v_{j,k} \) among the utility collection \( B_{j,k} \), \( u_{j,k} \) is the utility corresponding to choice \( C_j \) and outcome \( O_k \in A_k \), and \( n_k \) is the cardinality of focal element \( A_k \). The result of formula (2), \( E(C_j) \), is called the generalised expected utility of choice \( C_j \).

We can prove that the DS-IGOWA operator has some nice properties as follows:

**Theorem 1.** Let \( M \) be a DS-IGOWA operator. Then:

(i) **Commutativity:** \( M((v_{1,1}, u_{1,1}), \cdots, (v_{n,h}, u_{n,h})) = M((u_{1,1}, v_{1,1}), \cdots, (u_{n,h}, v_{n,h})) \)

(ii) **Monotonicity:** \( \forall j \), \( u_{j,k} \geq u_{i,k} \Rightarrow M((v_{1,1}, u_{1,1}), \cdots, (v_{n,h}, u_{n,h})) \geq M((v_{1,1}, u_{1,1}), \cdots, (v_{n,h}, u_{n,h})) \)

(iii) **Idempotency:** \( \forall j \in \{1, \cdots, n\} \), \( u_j = a \Rightarrow M((v_{1,1}, u_{1,1}), \cdots, (v_{n,h}, u_{n,h})) = a \)

(iv) **Bounds:** \( \min\{u_i\} \leq M((v_{1,1}, u_{1,1}), \cdots, (v_{n,h}, u_{n,h})) \leq \max\{u_i\} \)

**Proof:** We only need to check the property of commutativity; others can be checked similarly. By formula (2), we have:

\[
M((v_{1,1}, u_{1,1}), \cdots, (v_{1,k}, u_{1,k}), \cdots, (v_{n,h}, u_{n,h})) = \sum_{k=1}^{h} m(A_k) \left( \sum_{i=1}^{n_k} w_i b_{i,k}^\lambda \right)^{1/\lambda},
\]

\[
M((v_{1,1}, u_{1,1}), \cdots, (v_{1,k}, u_{1,k}), \cdots, (v_{n,h}, u_{n,h})) = \sum_{k=1}^{h} m(A_k) \left( \sum_{i=1}^{n_k} w_i b_{i,k}^\lambda \right)^{1/\lambda}.
\]

Since \((v_{1,1}^*, u_{1,1}^*), \cdots, (v_{n,h}^*, u_{n,h}^*)\) is any permutation of \((v_{1,1}, u_{1,1}), \cdots, (v_{n,h}, u_{n,h})\) for each focal element \( A_k \), we have \( b_{i,k}^* = b_{i,k} \). Then,

\[
M((v_{1,1}, u_{1,1}), \cdots, (v_{n,h}, u_{n,h})) = M((v_{1,1}^*, u_{1,1}^*), \cdots, (v_{n,h}^*, u_{n,h}^*)).
\]

### IV. Paradoxes Solving

In this section, we will validate our model by solving two well-known paradoxes: the Ellsberg paradox and the Machina paradoxes.

The Ellsberg paradox [22] is an important counter-example to the subjective expected utility (SEU) [23], [25]. It shows that most human decision makers prefer the choice with the precise probabilities to the choice with the imprecise probabilities [22], [25]. This phenomenon is called ambiguity aversion [22], [25]. Then, to be consistent with the Ellsberg paradox, many researchers have proposed many models to remedy SEU [25]. One of these models is Choquet expected utility (CEU) [28]. Unfortunately, Machina [23] further proposes two examples (i.e., the Machina paradoxes) to challenge the CEU. Actually, Machina paradoxes are not only the paradox of CEU but also the paradox of other models of handling ambiguity aversion [25].

**A. Solving the Ellsberg Paradox**

The Ellsberg paradox [22] is shown in Table I. In the experiment, there is an urn known to contain 30 red balls and 60 black and yellow balls, i.e., the proportions between the black ball and the yellow ball are unknown. In choices \( C_1 \) and \( C_2 \), one ball is to be drawn randomly from the urn. As shown in Table I, choice \( C_1 \) is a bet on red and choice \( C_2 \) is a bet on black. The decision maker has to choose one between \( C_1 \) and \( C_2 \). In the same circumstances, choice \( C_3 \) is a bet on red or yellow and choice \( C_4 \) is a bet on black or yellow as shown in Table I. The decision maker has to choose one between \( C_3 \) and \( C_4 \). Ellsberg [22] discovers that although the SEU model implies that if \( C_1 \succ C_2 \) then \( C_3 \succ C_4 \), most human decision makers prefer \( C_1 \) to \( C_2 \) and prefer \( C_4 \) to \( C_3 \) in the experiment.

Now we use our model to handle the paradox. Clearly, there are two focal elements: \( A_1 = \{O_1\} \) and \( A_2 = \{O_2, O_3\} \), where \( O_1, O_2, \) and \( O_3 \) stand for the red ball, the black ball, and the yellow ball, respectively. Thus, we have a mass function \( m: m(A_1) = \frac{30}{90} = \frac{1}{3} \) and \( m(A_2) = \frac{60}{90} = \frac{2}{3} \). And the utility collection is \( B_{j,k} = \{(v_{j,n}, u_{j,n}) | O_k \in A_k\} \), where \( O_k \) is the outcome in focal element \( A_k \). When the choice is \( C_j, u_{j,k} \) is the utility of \( O_k \) with the corresponding inducing variable \( v_{j,n} \), which is very clear as shown in Table I. Now suppose each inducing variable \( v_{j,n} \) be equal to the corresponding utility \( u_{j,n} \), i.e., \( v_{j,n} = u_{j,n} \). For example, as shown in Table I, focal element \( A_1 = \{O_1\} \) has only one element, and when the decision maker makes a choice of \( C_1 \) and the outcome is \( O_1 \) (red ball), the utility is 100; focal element \( A_2 = \{O_2, O_3\} \) has two elements, and when the decision maker selects choice \( C_1 \) and the outcome is \( O_2 \) or \( O_3 \) (black ball or yellow ball), the utility is 0 for both situations. So, if the choice is \( C_1 \), then \( B_{11} = \{(100, 100)\} \) and \( B_{12} = \{(0, 0), (0, 0)\} \). We set

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>red</td>
<td>black</td>
<td>yellow</td>
<td>red</td>
</tr>
<tr>
<td>$100$</td>
<td>$0$</td>
<td>$100$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$100$</td>
<td>$0$</td>
<td>$100$</td>
</tr>
</tbody>
</table>

**TABLE I.** ELLSBERG PARADOX.
Therefore, we have \( C_4 \succ C_2 \) and \( C_3 \prec C_4 \).

Thus, our model can capture the ambiguity aversion of the decision maker.

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### B. Solving the Machina Paradoxes

The first example of Machina paradoxes [23] is shown in Table II. CEU implies that if \( C_1 \succ C_2 \) then \( C_3 \succ C_4 \) [23]. However, Machina [23] points out that they should be \( C_1 \succ C_2 \) and \( C_3 \prec C_4 \). In this example, there is an urn holding 101 balls. Each ball is labelled with a number in \( \{1, \ldots, 4\} \). The decision maker knows that there are 50 balls labelled with either 1 or 2, and 51 balls labelled with either 3 or 4, but he is not sure which number it is [23]. As shown in Table II, the decision maker knows that the probability of being either 1 or 2 (i.e., \( E_1 \) or \( E_2 \)) is 50/101, and the probability of being either 3 or 4 (i.e., \( E_3 \) or \( E_4 \)) is 51/101. However, the decision maker does not know the probability of any given number being drawn. Then, the decision maker has to choose one between choice \( C_1 \) and choice \( C_2 \) or between choice \( C_3 \) and choice \( C_4 \).

Now, we use our model to solve this paradox. Clearly there are two focal elements: \( A_1 = \{O_1, O_2\} \) and \( A_2 = \{O_3, O_4\} \), where \( O_1, O_2, O_3, \) and \( O_4 \) stand for \( E_1, E_2, E_3, \) and \( E_4 \), respectively. Thus, we have a mass function \( m: m(A_1) = \frac{50}{101} \) and \( m(A_2) = \frac{51}{101} \). The inducing variables are shown in Table III. From these induced variables, we can see that the decision maker pays more attention on the utility of 0 than others. Now we set weighting vector \( W_2 = (0.25, 0.75) \) to aggregate two utilities into one in each focal element, and \( \lambda = 1 \) in the DS-IGOWA operator. The utility collection, \( B_{j,k} = \{(v_{j,n}, u_{j,n}) \mid O_n \in A_k\} \), is shown very clearly in Table IV. For example, focal element \( A_1 = \{O_1, O_2\} = \{E_1, E_2\} \) and when the decision maker selects choice \( C_1 \), the utility of the outcome is 8,000 for both \( E_1 \) and \( E_2 \); focal element \( A_2 = \{O_3, O_4\} = \{E_3, E_4\} \) and when the decision maker selects choice \( C_3 \) and choice \( C_4 \) and the utility of the outcome is 4,000 for both \( E_3 \) and \( E_4 \). So, if the choice is \( C_1 \), then \( B_{11} = \{(2,8000), (2,8000)\} \) and \( B_{12} = \{(1,4000), (1,4000)\} \). For each choice, we can calculate the generalised expected utility \( E(C_j) \) by formula (2). For example, we have:

\[
E(C_1) = m(A_1) \times U_{1,1}(B_{1,1}) + m(A_2) \times U_{1,2}(B_{1,2})
\]

Thus, we can easily prove:

\[
E(C_1) - E(C_2) > 0 \Rightarrow E(C_1) > E(C_2),
\]

\[
E(C_3) - E(C_4) < 0 \Rightarrow E(C_3) < E(C_4).
\]

Therefore, we have \( C_1 \succ C_2 \) and \( C_3 \prec C_4 \).

---

**Table I.** The First Example of Machina Paradoxes

<table>
<thead>
<tr>
<th>Choice</th>
<th>50 balls</th>
<th>51 balls</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( E_1 )</td>
<td>( E_2 )</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>$8,000</td>
<td>$8,000</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>$8,000</td>
<td>$4,000</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>$12,000</td>
<td>$8,000</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>$12,000</td>
<td>$4,000</td>
</tr>
</tbody>
</table>

**Table II.** The Inducing Variables in the First Example of the Machina Paradoxes

<table>
<thead>
<tr>
<th>( w_{2,1} )</th>
<th>( w_{2,2} )</th>
<th>( v_{j,n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

**Table III.** The First Example of Machina Paradoxes

<table>
<thead>
<tr>
<th>Choice</th>
<th>50 balls</th>
<th>51 balls</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( E_1 )</td>
<td>( E_2 )</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>$4,000</td>
<td>$8,000</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>$4,000</td>
<td>$4,000</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>$0</td>
<td>$8,000</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>$0</td>
<td>$4,000</td>
</tr>
</tbody>
</table>

**Table IV.** The Machina Paradoxes with Reflection
“objective advantage” over $C_1$ (respectively, $C_3$) because the urn has a 50:51 ball count [23]. Second, $C_1$ is unambiguous, but $C_2$ is ambiguous. So, the decision maker may feel that $C_2$’s ambiguity offsets its slight objective advantage, and prefer $C_1$ [23]. However, the ambiguous situation is unclear between $C_3$ and $C_4$. So, the decision maker may feel that this smaller ambiguity deference does not offset $C_4$’s objective advantage, and prefer $C_4$. Then, this is a trade-off between the objective advantage and the ambiguity attitude. To model this situation successfully, a model should capture the decision maker’s ambiguity aversion (it influences $C_1 > C_2$ mainly) and reflect the objective advantage (it influences $C_3 < C_4$ mainly). On the one hand, in our setting, $w_{2_1} = 0.25 < 0.5$. By Theorem 2, this model can capture the ambiguity aversion in the situation similar to the Ellsberg paradox. Then our model can reflect $C_1 > C_2$. On the other hand, since only the induced variable of utility $80$ is different from the argument in our setting, our model can put particular emphasis on the objective advantage in $C_3$ and $C_4$ by giving $4,000$ and $8,000$ bigger weighted value (if $w_{2_2} < 0.5$, then $w_{2_2} > 0.5 > w_{3_1}$).

We can generalise the above result. Let $u(x)$ represent the utility of $x (x \in \{4000, 8000, 12000\})$ with the corresponding inducing variable $v(x)$, and $u(0) = 0$ represent the utility of $0$ with the corresponding inducing variable $v(0)$. Then we have:

**Theorem 3.** In the first example of the Machina Paradox with the DS-IGOWA operator, if $w_{2_1} < m(A_1) < 0.5$ in the weighting vector $W_2 = (w_{2_1}, w_{2_2})$, and $v(x) = u(x) (x > 0)$, and $v(0) > \max\{u(x) \mid (x > 0)\}$, then $C_1 > C_2$ and $C_3 < C_4$.

**Proof:** By formula (2), we have:

$$E(C_1) = m(A_1) \times U_{1,1}(B_{1,1}) + m(A_2) \times U_{1,2}(B_{1,2})$$
$$= m(A_1) \times \sum_{i=1}^{2} w_{1_i} b_{i_1} + m(A_2) \times \sum_{i=2}^{2} w_{2_i} b_{i_2}$$
$$= m(A_1) \times (w_{2_1} \times u(8000) + w_{2_2} \times u(8000))$$
$$+ m(A_2) \times (u(2_1 \times u(4000) + w_{2_2} \times u(4000)))$$
$$= m(A_1) \times u(8000) + m(A_2) \times u(4000).$$

Similarly, by formula (2), we can get:

$$E(C_2) = u(8000) \times w_{2_1} + u(4000) \times w_{2_2},$$
$$E(C_3) = m(A_1) \times (u(12000) \times w_{2_1} + u(8000) \times w_{2_2})$$
$$+ m(A_2) \times u(4000) \times w_{2_2},$$
$$E(C_4) = m(A_1) \times (u(12000) \times w_{2_1} + u(4000) \times w_{2_2})$$
$$+ m(A_2) \times u(8000) \times w_{2_2}.$$

Thus, we can easily prove:

$$E(C_1) - E(C_2) > 0 \Rightarrow E(C_1) > E(C_2),$$
$$E(C_3) - E(C_4) < 0 \Rightarrow E(C_3) < E(C_4).$$

Therefore, we have $C_1 > C_2$ and $C_3 < C_4$.

Moreover, the Machina paradox with reflection [23] is shown as Table IV. Machina [23] points out that if $C_5 \succ C_6$ then $C_7 \prec C_8$, or if $C_5 \prec C_6$ then $C_7 \succ C_8$ because $C_5$ (respectively, $C_6$) is an informationally symmetric left-right reflection of $C_8$ (respectively, $C_7$). In this example, there is an urn holding 100 balls. Each ball is labelled with a number in \{1, 2, 3, 4\}. The decision maker knows that there are 50 balls labelled with either 1 or 2, and 50 balls labelled with either 3 or 4, but he is not sure which it is [23]. As shown in Table IV, the decision maker knows that with the probability of 100/100 it is either 1 or 2 (i.e., $E_1$ or $E_2$), and with the probability of 50/100 it is either 3 or 4 (i.e., $E_3$ or $E_4$). However, the decision maker does not know the probability of any given number being drawn. Then, the decision maker has to choose one between choice $C_5$ and choice $C_6$ or between choice $C_7$ and choice $C_8$.

Now we use our model to solve this paradox. Clearly there are two focal elements: $A_1 = \{O_1, O_2\}$ and $A_2 = \{O_3, O_4\}$, where $O_1, O_2, O_3,$ and $O_4$ stand for $E_1, E_2, E_3,$ and $E_4$, respectively. Thus, we have a mass function $m: m(A_1) = \frac{50}{100}$ and $m(A_2) = \frac{50}{100}$. The inducing variables are shown in Table V. From these induced variables, we can see that the decision maker pays more attention on the utility of 0 than others, which is the same as the one in the first example. Now we set weighting vector $W_2 = (0.25, 0.75)$ to aggregate two utilities into one in each focal element, and $\lambda = 1$ in the DS-IGOWA operator. The utility collection, $B_{1,2} = \{(v_{1,1}, u_{1,1}) \mid O_1 \in A_2\}$, is very clear as shown in Table IV. For example, as shown in Table IV, focal element $A_1 = \{O_1, O_2\} = \{E_1, E_2\}$ and when the decision maker selects choice $C_1$, the utility of the outcome is 4,000 for $E_1$ or 8,000 for $E_2$; focal element $A_2 = \{O_3, O_4\} = \{E_3, E_4\}$ and when the decision maker selects choice $C_1$ and the utility of the outcome is 4,000 for $E_3$ or 0 for $E_4$. So, if the choice is $C_1$, then $B_{1,1} = \{(1, 4000), (2, 8000)\}$ and $B_{1,2} = \{(1, 4000), (4, 0)\}$. For each choice, we can calculate the generalised expected utility $E(C_i)$ by formula (2). For example, the utility of choice $C_5$ is calculated as follows:

$$E(C_5) = m(A_1) \times U_{1,1}(B_{1,1}) + m(A_2) \times U_{1,2}(B_{1,2})$$
$$= m(A_1) \times \sum_{i=1}^{2} w_{1_i} b_{i_1} + m(A_2) \times \sum_{i=2}^{2} w_{2_i} b_{i_2}$$
$$= \frac{50}{100} \times (0.25 \times 8000 + 0.75 \times 4000)$$
$$+ \frac{50}{100} \times (0.25 \times 0.75 \times 4000)$$
$$= 4000.$$
Similarly, by formula (2), we can get:

$$E(C_5) = m(A_1) \times U_{1,1}(B_{1,1}) + m(A_2) \times U_{1,2}(B_{2,2}) = m(A_1) \times \sum_{i=1}^{5} w_i b_i + m(A_2) \times \sum_{i=1}^{2} w_i b_i = m(A_1) \times (w_{2,1} \times u(8000) + w_{2,2} \times u(4000)) + m(A_2) \times (w_{2,1} \times u(0) + w_{2,2} \times u(4000)) = m(A_1)w_{2,1}u(8000) + w_{2,2}u(4000).$$

Similarly, by formula (2), we can get:

$$E(C_6) = m(A_2)w_{2,2}u(8000) + m(A_1)u(4000),$$
$$E(C_7) = m(A_1)w_{2,1}u(8000) + m(A_2)u(4000),$$
$$E(C_8) = m(A_2)w_{2,1}u(8000) + w_{2,2}u(4000).$$

Thus, we can easily prove:

$$E(C_5) = E(C_6) < 0 \Rightarrow E(C_5) < E(C_6),$$
$$E(C_7) = E(C_8) > 0 \Rightarrow E(C_7) > E(C_8).$$

Therefore, we have $C_5 < C_6$ and $C_7 > C_8$.

In summary, our model successfully solves the Ellsberg paradox and the Machina paradoxes.

V. CONCLUSION

In this paper, we apply the IGOWA operator into D-S theory and propose a new decision rule called the DS-IGOWA operator, which can be used to make a decision under ambiguity well. In fact, using our model we can successfully solve the Ellsberg paradox and the Machina paradoxes. In the future, we will reveal some insights into our aggregation operators and apply our decision rule into game theory.

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